Written test Statistics for Electronics (Tuesday group 17.05)

Task 1: Given a 3 element set $\Omega = \{x, y, z\}$

a) equipp Ω with the structure of a probability space by assigning probabilities to the elementary events.

b) Define a random variable X on the probability space Ω (from task a) such that X has expectation zero and variance one.

c) Find two independent events A and B on Ω.

d) Find two disjoint events C and D on Ω.

Task 2: given a continuos random variable X with values in $[0, \infty)$ and density $f_X(z) = \left\{ \begin{array}{c} 3e^{-3z} \text{ for } z \geq 0 \\ 0 \text{ for } z < 0 \end{array} \right|$.
Compute the following:

a) The cumulative distribution function F of the random variable X and sketch the graphs of F and f

d) The probability $P\{-1 \leq X \leq 1\}$ that the random variable has value in the interval $[-1, 1]$.

c) The probability $P\{-2 \leq X \leq -1\}$ that the random variable has value in the interval $[-2, -1]$

d) The probability $P\{X = 5\}$

Task 3: Given a sequence of iid random variables $\{X_i\}_{1=1}^n$ with $X_i \in \{-1, +1\}$ and $P(X_i = 1) = \frac{1}{2}$

a) Use the central limit theorem to give an approximate upper bound on the probability that $\sum_{i=1}^{90000} X_i \geq 600$. List of values for the cumulative distribution function $\Phi(z)$ of the standard normal distribution (expectation zero and variance 1):

$\Phi(1) = 0.841\,34; \Phi(1.5) = 0.933\,19; \Phi(2) = 0.977\,25;\ \Phi(2.5) = 0.993\,79; \Phi(3) = 0.998\,65; \Phi(3.5) = 0.999\,77; \Phi(4) = 0.999\,97$

b) (optional): Use the Berry- Esseen theorem to estimate the maximal error made in the approximate solution to the previous task 7a

Hint : The Berry-Esseen theorem states the following: Given a sequence of iid random variables $\{Y_i\}_{i=1}^n$ with $\mathbb{E}Y_i = 0$, $Var(Y_i) = \sigma^2$ and $\rho = \mathbb{E}\left(|Y_i|^3\right)$. Let further F_n be the cumulative distribution function (cdf) of $\frac{1}{\sigma\sqrt{n}}\left(\sum_{i=1}^n Y_i\right)$ and let Φ be the cdf of the standard normal distribution $\mathcal{N}(0,1)$. Then the following inequality holds for every $n \geq 1$:

$$\sup_x |F_n(x) - \Phi(x)| \leq \frac{1}{2} \cdot \frac{\rho}{\sigma^3\sqrt{n}} \tag{1}$$

Task 4: Given two random variables $X \geq 0$ and $Y \geq 0$ with $EX = 1$, $EY = 2$ and $VarX = 1$, $VarY = 3$

a) Use the Markov inequality to estimate $\Pr\{3X + 2Y \geq 70\}$

b) Compute the variance of $2X - 2Y$ under the assumption that X and Y are independent

Task 5 : Given 5 independent samples of a random variable X . The sample values are $\{1; 3; 2; 1; 4\}$. It is known that the random variable X has a density of the form $\varphi(x) = ae^{-ax}$ for $x \geq 0$ and zero otherwise . The parameter $a > 0$ is unknown . Find the maximum likelihood estimator $\tilde{a}_{ML}$ of the parameter a with respect to the above sample values.